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Recovery of Dirac Equations from Their Solutions[1]

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We deal with quantum field theory in the restriction to external Bose fields. Let $(i\gamma^\mu \partial_\mu - \mathcal{B})\psi = 0$ be the Dirac equation. We prove that a nonquantized Bose field $\mathcal{B}$ is a functional of the Dirac field ψ whenever this ψ is canonical. Performing the verification for $\mathcal{B} := m =$ const, which yields the free Dirac field, we also prepare the tedious verifications for all $\mathcal{B}$ which are nonquantized and static. Such verifications must not be confused, however, with the proof of our formula, which is shown in detail.

1. DIFFERENT DIRAC THEORIES

A canonical Dirac field ψ is never an operator, but an operator-valued distribution (Jost 1965). For such a four-component spinor, its Hermitian conjugate $\psi^\dagger$, and its transpose ψ^T, we postulate the anticommutators

$$\begin{aligned} [\psi(x), \psi(y)^\dagger]_+ \, \delta(x^0 - y^0) &= \delta(x - y), \\ [\psi(x), \psi(y)^T]_+ \, \delta(x^0 - y^0) &= 0 \end{aligned} \tag{1}$$

where $\delta(z) := \delta(z^0)\delta(z^1)\delta(z^2)\delta(z^3)$. We consider the Dirac equation

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$$\{i\,\partial\!\!\!/^x - \mathcal{B}(x)\}\psi(x) = 0 \qquad \text{with} \quad \partial\!\!\!/^x := \gamma^\mu\partial/\partial x^\mu, \quad \mathcal{B} = \gamma_0\mathcal{B}^\dagger\gamma_0 \tag{2}$$

[the latter makes the action $\int\overline{\psi}(i\partial\!\!\!/ - \mathcal{B})\psi$ Hermitian]. Evidently, $\mathcal{B}$ is a member of Dirac's Clifford algebra Cl_D. We need the time-ordered product

$$b(x, z) := (4\pi)^2\, T\psi(x + z)\overline{\psi}(x - z) \in Cl_D \tag{3}$$

The covariant ordering needed here must directly act only on basic fields, not on their derivatives. Hence the time ordering of the latter must be *defined* by

$$T\phi_{,\mu}(x)\chi(y) := \partial^x_\mu\, T\phi(x)\chi(y)$$

This widely used, but rarely emphasized prescription has been explained by Nambu (1952, Eq. (3.9)), Callan *et al.* (1970, Eq. (2.9)), DeWitt (1984, p. 246), Just and The (1986, Appendix B), Sterman (1993, p. 114), and DeWitt-Morette (1994). Of the canonical relations (1), only the first will be used here, but both are needed to define ψ completely.

Further treatment of $\mathcal{B}$ and b can proceed in four ways, of which only the last will be pursued here:

(a) One may *desire* that $\mathcal{B}$ also be a canonical field. This gives the usual 'effective' field theory (Weinberg 1995–2000). One starts from (1) and (2) and their extensions to Bose fields, but all these break down under the infinite renormalization (Brandt 1969). Hence that desire, explained in the introductions of many books on quantum fields, is only satisfied as long as one does not admit interactions.

(b) All divergences are prevented in quantum induction (QI), where $\mathcal{B}$ is a noncanonical quantum field (Just and Sucipto, 1997). For this unconventional theory, peripheral results have been explained briefly, but only at the expense of setting aside the proofs (Just and Thevenot, 2000; Just *et al.*, 2000).

(c) Some divergences are also avoided when one restricts $\mathcal{B}$ to be nonquantized forever. This is done in the mathematical theory of heat kernels (Esposito 1998), where one studies elaborately the boundary conditions for (3) at large separations z.

(d) In this paper, we examine a simple consequence of the postulates (1) and (2). It also holds in (b), but now we prove it only for *nonquantized* $\mathcal{B}$ (for clarity excluded from QI); hence the present proof holds as well for (c). We nevertheless do not apply heat kernels, because 'outer' boundary conditions on (3) are superfluous here.

For case (d), we prove in Section 2 the explicit recovery of $\mathcal{B}$ from (2) as a functional of ψ. The result is verified for the constant $\mathcal{B} = m$ in Section 3. Restricting the nonquantized $\mathcal{B}$ to a static $\beta(\vec{x})$ in Section 4, we *prepare* its recovery in Section 5.

2. THE RECOVERY FORMULA

In what follows,

$$\not{z}_-^{-3} := (z^2 - i\epsilon)^{-2}\not{z} \qquad \text{with} \quad \epsilon \to +0 \tag{4}$$

For (3), the canonical postulates (1) and (2) imply

$$\{\not{\partial}^x + \not{\partial}^2 + 2i\mathcal{B}(x + z)\}b(x, z) = 2\pi^2\,\delta(z) = i\not{\partial}^z\not{z}_-^{-3} \tag{5}$$

Here we have used (4) in order to give to (3) the analyticity of a time-ordered product. At this point, it will be useful to introduce

$$r(x, z) := b(x, z) - i\not{z}_-^{-3} \tag{6}$$

We shall see that this *remainder* is less singular than $\not{z}_-^{-3}$ for $z \to 0$. Using (6) in (5), we obtain

$$2\mathcal{B}(x + z)\,\not{z}_-^{-3} = \{\not{\partial}^x + \not{\partial}^z + 2i\mathcal{B}(x + z)\}r(x, z) \tag{7}$$

$$\approx \not{\partial}^x r(x, z) = \not{\partial}^x[b(x, z) - i\not{z}_-^{-3}] \tag{8}$$

It is essential that the *equalities* in (7) and (8) hold strictly, whereas the left side of (8) only approximately equals the right side of (7). In (8), we have used that $z \to 0$ makes the remainder $r(x, z)$ singular, such that the strongest singularity on the right of (7) is contained in $\not{\partial}^z r(x, z)$. Comparing the left sides of (7) and (8), we see that $r(x, z)$ is less singular than $\not{z}_-^{-3}$, then we eliminate it by (6). The resulting $\not{\partial}^z\,\not{z}_-^{-3} = \text{const} \cdot \delta(z)$, however, drops out when we multiply (7) and (8) by $\not{z}^3$, giving

$$2\mathcal{B}(x) + \cdots = [\not{\partial}^z b(x, z) + \cdots]\not{z}^3 \tag{9}$$

The dots symbolize terms which we have neglected in (8) or in the approximation $\mathcal{B}(x + z) \approx \mathcal{B}(x)$. All these terms contribute nothing to (9) with $z \to 0$; hence

$$2\mathcal{B}(x) = \lim_{z\to 0}\,[\not{\partial}^z b(x, z)]\,\not{z}^3 \tag{10}$$

While (9) is a quantum field, its local limit (10) is nonquantized because we assumed this in (2). Noting (3), we see that (10) has proved

$$\mathcal{B}(x) = 8\pi^2 \lim_{z\to 0}\,[\not{\partial}^z T\psi(x + z)\,\overline{\psi}\,(x - z)]\,\not{z}^3 = \gamma_0\mathcal{B}(x)^\dagger\gamma_0 \tag{11}$$

The second assertion follows when we start from (5) with the differential operator replaced by one which acts on the bilocal field $b(x, z)$ from the *right* side.

In (11), the multiplication by $\not{z}^3 \to 0$ has removed the singularity. Therefore, the step functions in the time ordering need not be differentiated. Hence the $\not{\partial}^z$ can be expressed by operators acting on x, giving

$$\mathcal{B}(x) = 8\pi^2\gamma^\mu \lim_{z\to 0} T\psi(x - z) \overleftrightarrow{\partial}{}^x_\mu \overline{\psi} (x + z) z^3 \tag{12}$$

Here we need no longer indicate that no differentiation acts on z^3. Thus we have *recovered* the nonquantized Bose field with which Dirac's equation (2) has been solved, provided this has been done by a Dirac field ψ satisfying (1).

In this paper we ask to what extent (11) can be verified by two examples:

1. $\mathcal{B} = m =$ const, which yields the free ψ.
2. Static nonquantized $\mathcal{B}(x) := \beta(\vec{x})$.

Since neither of these examples is a quantum field, the assumptions of heat kernels are valid here (Esposito, 1998). For the free Dirac field, we verify in Section 3 the recovery of $\mathcal{B} = m$ by (11). For nonquantized and static $\mathcal{B}$, the complete solution ψ of (1) and (2) follows in principle from an eigenvalue problem in *three* dimensions. For such a case, we make the functional (11) more specific in Section 4. Sections 3–7 describe both a very easy and an extremely difficult verification of (11). Its rigorous proof [under the conditions of (d) in Section 1] is *completed* at (10).

3. A SIMPLE VERIFICATION

Let us define $\delta(p, q)$ such that the measure

$$d(p) = (2\pi)^{-3}\theta(p_0)\, \delta(p^2 - m^2)dp \tag{13}$$

over the sharp mass shell $p_0 = \sqrt{\vec{p}^{\,2} + m^2}$ makes

$$\int f(p)d(p)\delta(p, q) = f(q) \qquad \text{for} \quad q_0 = \sqrt{\vec{q}^{\,2} + m^2}$$

For $m = \text{const} > 0$, the nonquantized spinors $u_\sigma(p)$ with helicity label σ are to fulfill

$$(\not{p} \mp m)u^\pm_\sigma(p) = 0 \qquad \text{and} \qquad \sum_\sigma u^\pm_\sigma(p)\overline{u}^\pm_\sigma(p) = \not{p} \pm m$$

With the Poincaré invariant vacuum $|\,\rangle$, we postulate

$$a^\pm_\sigma(p)|\rangle = 0 \qquad \text{and} \qquad [a^\pm_\sigma(p), a^\pm_\tau(q)^\dagger]_+ = \delta_{\sigma\tau}\delta(p, q)$$

All other anticommutators of the $a^\pm_\sigma(p)$ are assumed to vanish. Then (1) and (2), with $\mathcal{B} = m =$ const, are satisfied by the free canonical Dirac field,

$$\psi(x) = \sum_\sigma \int \{e^{-ipx} u^\sigma_+(p)a^+_\sigma(p) + e^{+ipx}u^\sigma_-(p)\, a^-_\sigma(p)^\dagger\}\, d(p) \tag{14}$$

Its familiar propagator will be needed in the form

$$(2\pi)^4 \langle |T\psi(2z)\overline{\psi}(0)|\rangle = i \int e^{-2ipz}\, dp(\not{p} + i\epsilon - m)^{-1} \qquad (\text{with } \epsilon \to +0)$$
$$= \pi^2(i\, \not{z}_-^{-3} - mz_-^{-2} + \cdots) \tag{15}$$

Since (11) is nonquantized, it equals its expectation value in any state such as $|\ \rangle$. Hence (11) is *verified* by (15), because it yields

$$\mathcal{B}(x) = 8\pi^2 \lim_{z\to 0} \langle |\not{\partial}^z\, T\psi(2z)\overline{\psi}(0)|\rangle\, \not{z}^3 = m \tag{16}$$

No further solution of Dirac's equation (2) is known for which (11) can be verified as easily.

4. STATIC BACKGROUNDS

When $\mathcal{B}$ is not only nonquantized, but also time independent, we define

$$\beta(\vec{x}) := \mathcal{B}(x) \tag{17}$$

Let the spinors $u^\sigma(x)$ solve the eigenvalue problem

$$H\, u^\sigma = \omega_\sigma \cdot u^\sigma \qquad \text{with} \quad H := \gamma_0\{\beta(\vec{x}) - i\not{\partial}^x\} \tag{18}$$

Although we use $\not{\partial}^x = \gamma^\mu\, \partial/\partial x^\mu$ to avoid additional notations, (18) involves only $x^r \in \{x^1, x^2, x^3\}$ because β and u^σ are independent of x^0. Since (11) for (17) makes $\beta^\dagger\gamma_0 = \gamma_0\beta$, the operator H is *Hermitian*. Hence its eigenvalues ω_σ are real and the solutions can be made orthogonal:

$$\int u^\sigma(\vec{x})^\dagger\, d^3x\, u^\tau(\vec{x}) = \delta^{\sigma\tau} \qquad \text{for} \quad \sigma, \tau \in \{1, 2, \ldots\} \tag{19}$$

For brevity, we use notations suitable for a discrete frequency spectrum, although that of (18) will often be continuous as in (14) or mixed as in the hydrogen atom. In either case, the σ in (18) takes infinitely many values, in contrast to (14), where it labels two helicities. In addition, we assume

$$\sum_\sigma u^\sigma(\vec{x})u^\sigma(\vec{y})^\dagger = \delta^3(\vec{x} - \vec{y}) \tag{20}$$

This *completeness* relation will be most important here. It is compatible with (19), but not implied by this. Then Dirac's equation (2) is satisfied by each term of

$$\psi(x) = \sum_\sigma e^{-i\omega_\sigma x^0} u^\sigma(\vec{x})a_\sigma \tag{21}$$

We find (1) satisfied when we make (21) a quantum field by postulating

$$[a_\sigma, a_\tau^\dagger]_+ = \delta_{\sigma\tau} \qquad \text{and} \qquad [a_\sigma, a_\tau]_+ = 0 \tag{22}$$

5. DESIRABLE VERIFICATIONS

We specify a ground state $|\cdot\rangle$ by separating positive and negative frequencies in (21):

$$\psi(x) = \sum_\sigma e^{-i\omega_\sigma x^0} u_+^\sigma(\vec{x}) a_\sigma + \sum_\tau e^{+i\Omega_\tau x^0} u_-^\tau(\vec{x}) b_\tau^\dagger \tag{23}$$

$$a_\sigma|\cdot\rangle = 0 \qquad \text{and} \qquad b_\tau|\cdot\rangle = 0$$

where $\omega_\sigma > 0$ and $\Omega_\tau := -\omega_\tau > 0$. Rewriting the anticommutators (22) in the notation (23), we deduce the *propagator*

$$\begin{aligned} F(x, z) &:= \langle\cdot|T\psi(x+z)\overline{\psi}(x-z)|\cdot\rangle \\ &= \theta(z^0) \sum_\sigma e^{-2i\omega_\sigma z^0} u_+^\sigma(\vec{x}+\vec{z})\, \overline{u}_+^\sigma(\vec{x}-\vec{z}) \\ &\quad - \theta(-z^0) \sum_\tau e^{+2i\Omega_\tau z^0} u_-^\tau(\vec{x}+\vec{z}) \overline{u}_-^\tau(\vec{x}-\vec{z}) \end{aligned} \tag{24}$$

which, unlike (15), is not Poincaré covariant. Having restricted the field β in (17) to become nonquantized, we find it equal to its expectation value

$$\beta(\vec{x}) = \langle\cdot|\mathcal{B}(x)|\cdot\rangle = 8\pi^2 \lim_{z\to 0} [\not{\partial}^z F(x, z)]\, \not{z}^3 \tag{25}$$

Since (11) is the same as (12), we can in (25) with (24) omit those terms in which the $u_\pm^\sigma(\vec{x})$ are not differentiated. Returning from (23) to the compact notation (21), we obtain

$$\begin{aligned} \frac{1}{8\pi^2} \beta(\vec{x}) = \gamma^\tau \lim_{z\to 0} \sum_\sigma u^\sigma(\vec{x} - \vec{z})\, \overleftrightarrow{\partial}_r^x\, \overline{u}^\sigma(\vec{x}+\vec{z}) \\ \{\theta(-z_0)\theta(\omega_\sigma) - \theta(z_0)\theta(-\omega_\sigma)\} e^{2i\omega_\sigma z_0}\, \not{z}^3 \end{aligned} \tag{26}$$

In all the limits taken in (10)–(26), $z = 0$ may be approached on any line through Minkowski space which does not touch the cone $z^2 = 0$. Hence (26) can be specialized in many ways. Starting with $\vec{z} \equiv 0$, for instance, we see that the matrices $u^\sigma(\vec{x})\, \overleftrightarrow{\partial}_r^x\, \overline{u}^\sigma(\vec{x}) e^{2i\omega_\sigma z_0}$ must *increase* so strongly that their sums behave as $(z_0)^{-3}$ for $z_0 \to \mp 0$ and $\omega_\sigma \to \pm\infty$. Alternatively, we may start with $z_0 \equiv \mp 0$, so that (26) simplifies to

$$\frac{1}{8\pi^2}\,\beta(\vec{x}) = \gamma^r \lim_{z\to 0} \sum_\sigma \theta(\omega_\sigma) u^\sigma(\vec{x} - \vec{z}) \overleftrightarrow{\partial}{}^x_r\, \bar{u}^\sigma(\vec{x} + \vec{z})(\gamma_s z^s)^3 \tag{27}$$

$$= -\gamma^r \lim_{z\to 0} \sum_\sigma \theta(-\omega_\sigma) u^\sigma(\vec{x} - \vec{z})\, \overleftrightarrow{\partial}{}^x_r\, \bar{u}^\sigma(\vec{x} + \vec{z})(\gamma_s z^s)^3$$

Here, as in (23)–(26), the sum runs either over all solutions of (18) with frequencies $\omega_\sigma > 0$ or over those with $\omega_\sigma < 0$.

6. GENERAL REMARKS

In the Coulomb field of a proton, (24) results from all the spinors u^σ of either an electron or a positron. For their partly continuous spectra, suitable *notations* must be invented because we have for brevity used those for discrete ω_σ. In either case, however, the result must satisfy

$$\beta(\vec{x}) = m + \lambda_0 \frac{e}{|\vec{x}|} \tag{28}$$

Since the nonquantized and static fields (17) include the $\mathcal{B} = m$ of the free Dirac field (14), the $\beta = m$ must also follow from (26). However, verifying this will be more difficult than under the manifest Lorentz covariance employed in Section 3. The greatest obstacle to any use of (26) is that it requires *infinitely* many exact solutions of (18).

Thus we have performed one of those verifications which are possible as indicated in Section 5 [namely that of (28) with $e = 0$], but we did so in a much simpler way. The verification shown in Section 3 consists of the *single* line (16) because (13)–(15) merely state our notations. Having tried to evaluate (26) for (18) with $\beta(\vec{x}) = m$, we know that doing so will cost much work. Hence that attempt has shown that a problem which under Lorentz covariance is trivial can be poorly tractable when this is not manifest.

All this does not concern a physical theory. It rather forms a didactic *simplification* (by nonquantized Bose fields) of a mathematical result from QI. This new version of quantum field theory has only recently been suggested (Just and Sucipto, 1997). Hence the proof of (12) for quantum fields $\mathcal{B}$ must be deferred until publication of QI.

7. RESULTS AND EXPECTATIONS

Whereas (28) provides one of the few simple problems in which all solutions of (18) are known, (26) must hold for every $\beta(\vec{x})$ admitted here. For known as well as unknown u^σ, we thus obtain the following result:

Recovery Theorem. Whenever the solutions $u^\sigma(\vec{x})$ of Dirac's equation (18) with any nonquantized and time-independent matrix $\beta(\vec{x}) \in Cl_D$ fulfill the completeness relation (20), that field $\beta(\vec{x})$ is recovered by (26).

Comparing this result with the inverse scattering theory (Bertero and Pike 1992), we see that in some respect the opposite is done there. One wants to derive *approximations* to a potential by using as few as possible of its consequences. On the contrary, we recover $\beta(\vec{x})$ exactly by (26), but only when the exact solutions $u^\sigma(\vec{x})$ of (18) are known (either for all $\omega_\sigma > 0$ or for all $\omega_\sigma < 0$). The further analysis of (5) reveals that (12) must satisfy consistency conditions, such as Dirac induced field equations and the absence of Pauli terms (Just and Thevenot 2000), but these do not invalidate the present results.

In our derivation, we used quantum field theory (Jost, 1965) in the restriction to external Bose fields (Esposito, 1998). However, the resulting 'solution' of (18) with the Bose field (17) does not involve quantum fields and not even time coordinates. Thus it should equally well be of interest to readers who treat in Dirac's equation (2) not only the matrix $\mathcal{B}$, but also the spinor ψ as *nonquantized* fields (Thaller, 1992). For this case [in which (1) is ignored], our general result (12) might not be needed if one merely wants to derive (26) from (18)–(20), hence without (21)–(25). Thus there remains the following:

Question. Is there a simpler way to prove (26), or will our approach remain the best method to reach that result about classical solutions of the time-independent Dirac equation (18)?

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